Coherence and stability in large-scale networks with distributed dynamic feedback

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ACCESS the future

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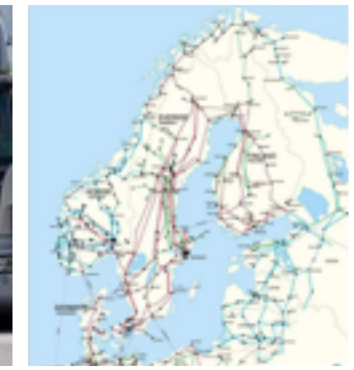
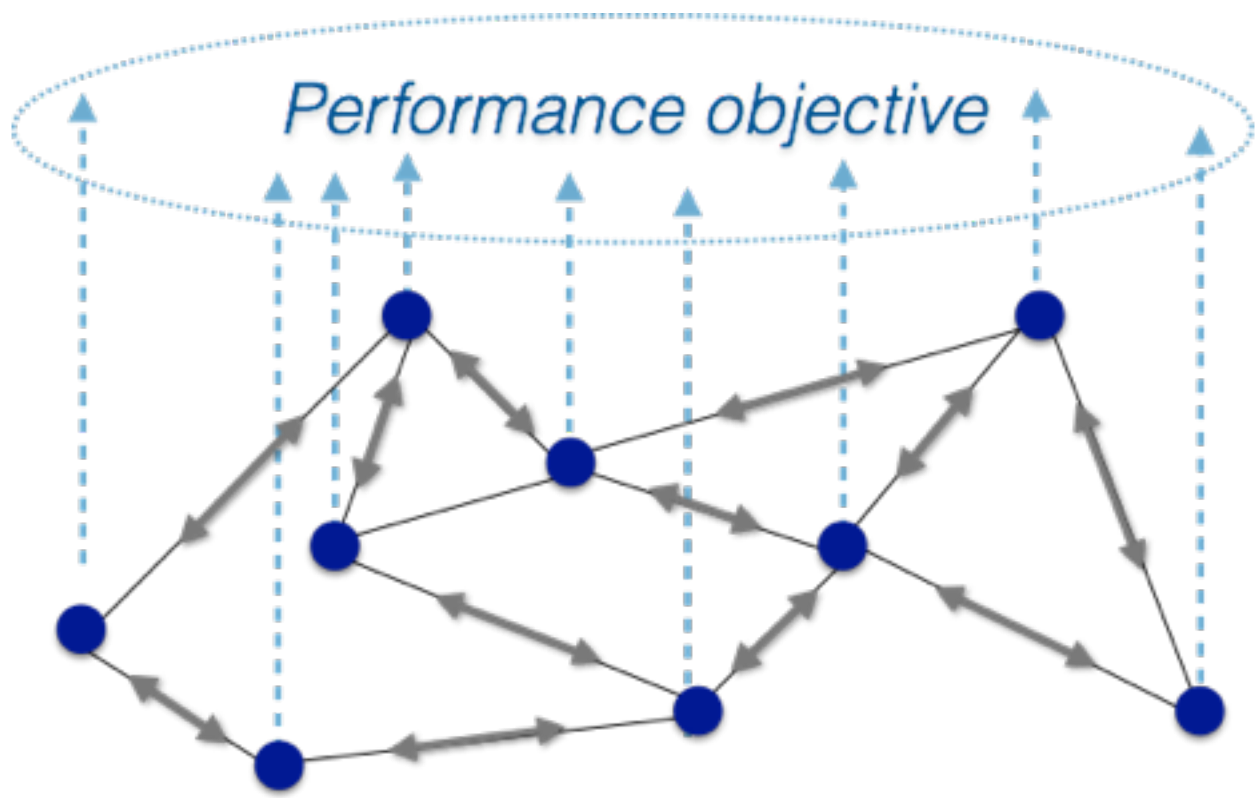
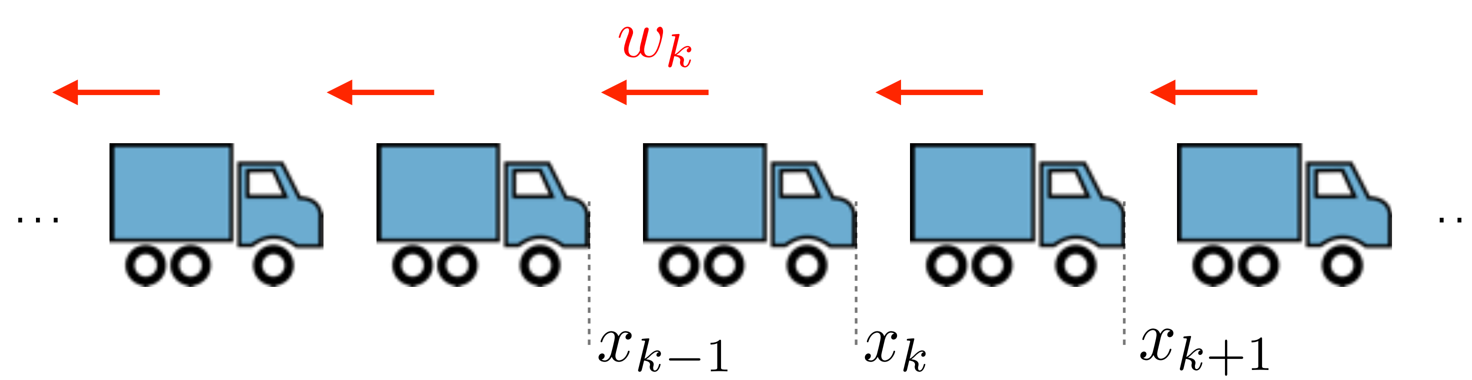
Networked dynamical systems: *global* objectives, but *local* feedback



Are there *fundamental limitations* to network *performance*?

Vehicle platoons: can global performance be ensured under disturbances?

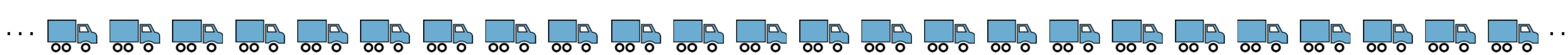


- Global control objectives:
  - common cruising speed $\bar{v}$
  - tight constant spacing Δ
- Dynamics (example): look-ahead, look-behind control

$$\ddot{x}_k = \dot{v}_k = f_+(x_{k+1} - x_k - \Delta) + f_-(x_{k-1} - x_k - \Delta) + g_+(v_{k+1} - v_k) + g_-(v_{k-1} - v_k) + w_k$$

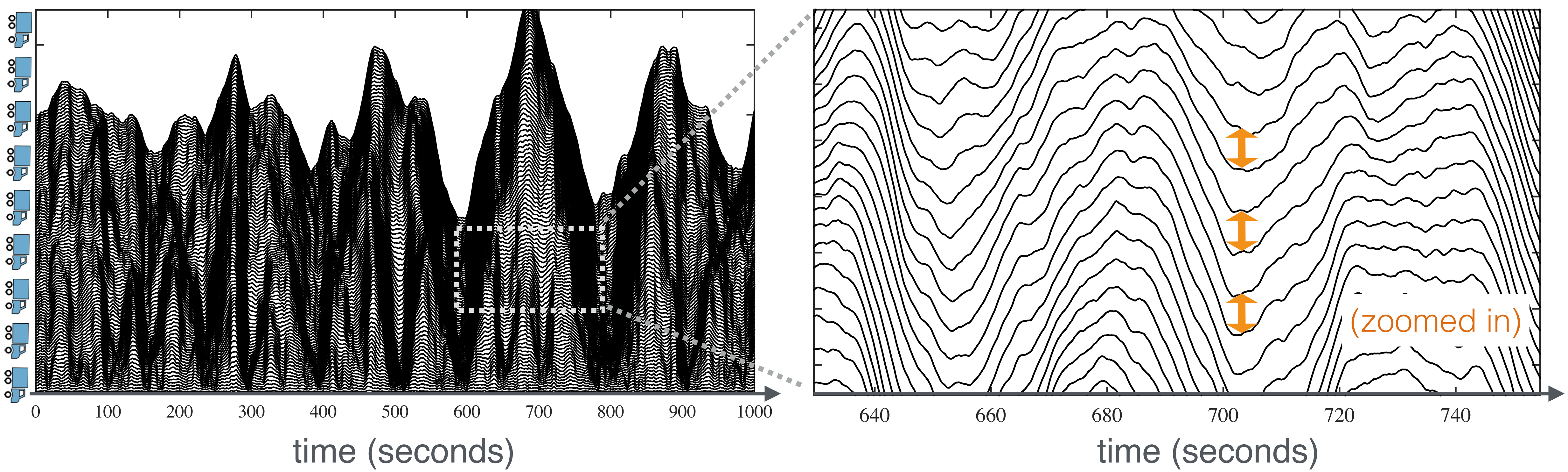
(f_+, f_-, g_+, g_- constant gains)

- With disturbances: objectives only achieved approximately
- What happens if the platoon grows?



Performance issues if control is based on *relative* measurements

Time trajectories of 100 vehicles, relative to leader, seen from above



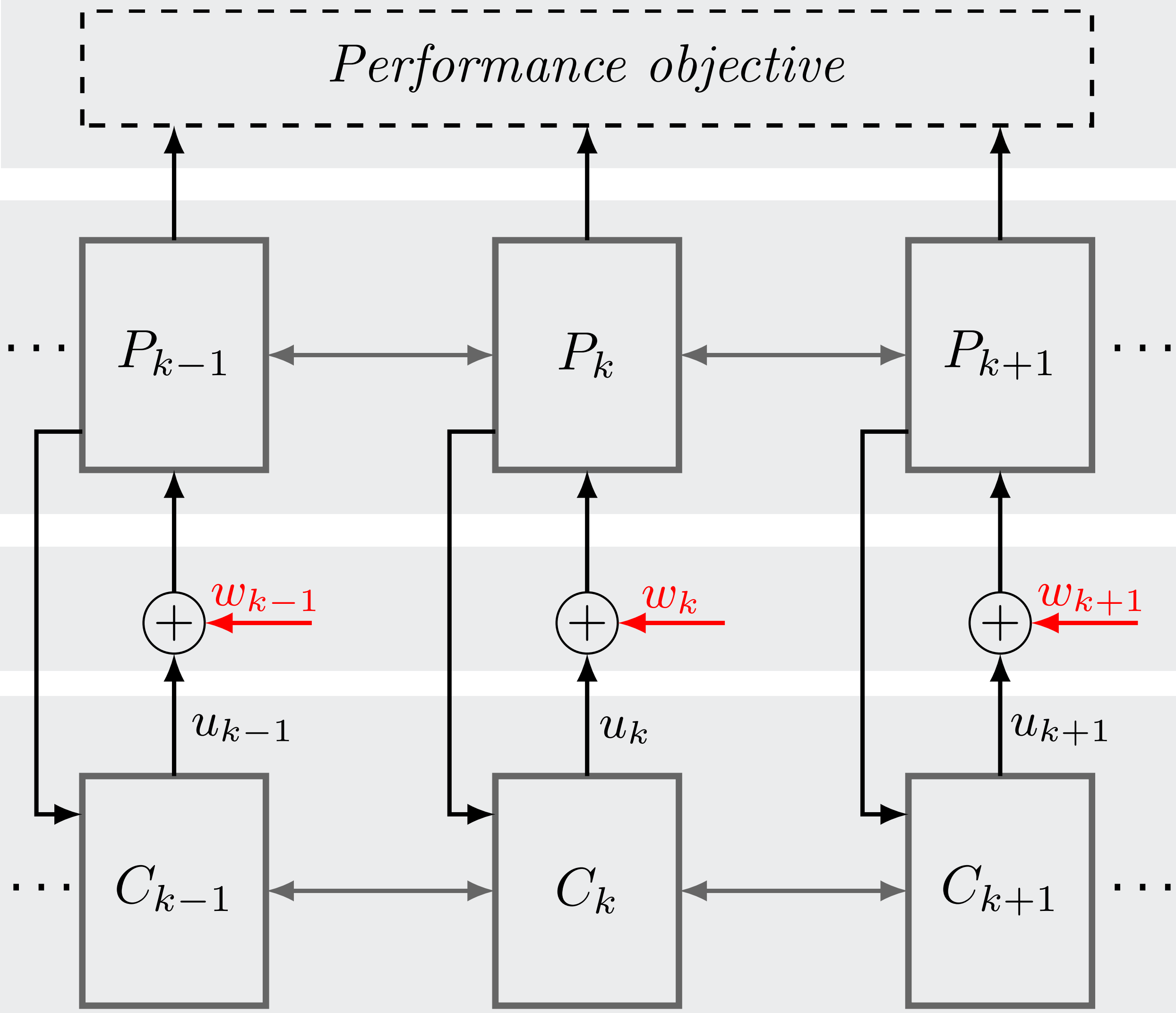
- Formation is stable
- Spacings ⟷ are well-regulated (no collisions, no string instability!)
- However - not a *rigid* formation, not *coherent!*
- Fundamental limitation to local, static feedback (Bamieh *et al.*, 2012)

Can *dynamic* control laws help?

Setup: Evaluating performance of distributed control systems

- Performance: measure of *coherence* (deviation from network average)

$$V_k = \mathbb{E}\left\{\left(x_k - \frac{1}{N}\sum_{j=1}^{N} x_j\right)^2\right\}$$



- Plant: Single / double integrator

$$\dot{x}_k = u_k + w_k \;\; / \;\; \ddot{x}_k = u_k + w_k$$

- Additive white noise

- Control law: consensus dynamics

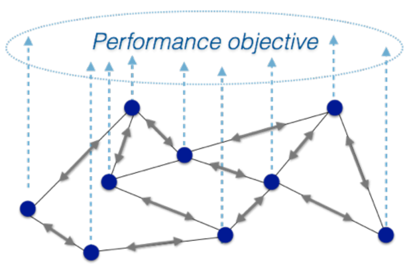
$$u_k = \sum_{j \in \mathcal{N}_k} f_{jk}(x_j - x_k) + \sum_{j \in \mathcal{N}_k} f_{jk}(\dot{x}_j - \dot{x}_k)$$

Relative feedback

$$-f_o x_k - g_o \dot{x}_k$$

Absolute feedback (/self-damping)

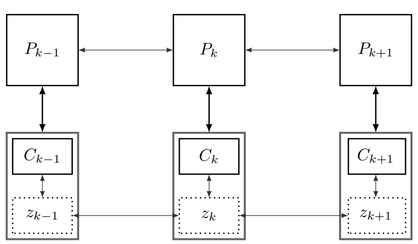
OUTLINE



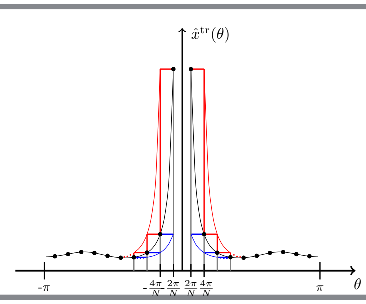
Introduction and problem overview

$\mathcal{H}_2$

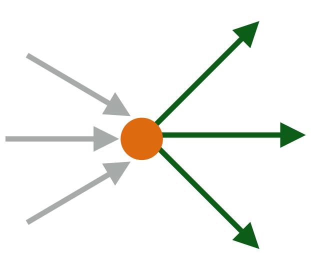
Evaluating input-output performance



Problem setup: static vs. dynamic feedback



Deriving asymptotic performance scalings



Conclusions and future work

Performance is evaluated through input-output H_2 norms

Consider general linear system under white noise input

$$\dot{x} = \mathcal{A}x + \mathcal{B}w \qquad (1)$$
$$y = \mathcal{C}x$$

Recall:
Need to evaluate $V_k = \mathbb{E}\{(y_k)^2\}$, with $y_k = x_k - \frac{1}{N}\sum_{j=1}^{N} x_j$.

Lemma:
The squared H_2 norm of (1) from input w to output y gives

$$||H||_2^2 = \lim_{t\to\infty} \mathbb{E}\{y^*(t)y(t)\},$$

That is, the steady state output variance.

➡ With the appropriate output, performance is given by H_2 norm!

Evaluating system performance amounts to evaluating H_2 norms!

Unitary transformation simplifies H_2 norm evaluation 1(2)

$$\hat{H}: \quad \dot{\hat{x}} = \underbrace{\begin{bmatrix} \ddots & \ddots \\ \ddots & \ddots \end{bmatrix}}_{\hat{\mathcal{A}}_{N\times N}} \hat{x} + \underbrace{\begin{bmatrix} \ddots \\ \ddots \end{bmatrix}}_{\hat{\mathcal{B}}} \hat{w}$$

$$\hat{y} = \underbrace{\begin{bmatrix} \ddots & \ddots \end{bmatrix}}_{\hat{\mathcal{C}}} \hat{x}$$

$$\hat{H}_n: \quad \begin{aligned} \dot{\hat{x}}_n &= \hat{\mathcal{A}}_n \hat{x}_n + \hat{\mathcal{B}}_n \hat{w}_n \\ \hat{y}_n &= \hat{\mathcal{C}}_n \hat{x}_n \end{aligned}$$

- Unitary transformation does not change H_2 norm
- (Block-) diagonalize to obtain N decoupled subsystems $\hat{H}_n$
- H2 norm is sum of subsystem norms:

$$||H||_2^2 = ||\hat{H}||_2^2 = \sum_{n=1}^{N} ||\hat{H}_n||_2^2$$

- *Zero mode* associated with *drift of average* makes $\hat{\mathcal{A}}_1$ non-Hurwitz.
- Mode *unobservable*, so $||\hat{H}_1||_2^2 = 0$.
- Only sum over remaining, stable, subsystems: $||H||_2^2 = \sum_{n=2}^{N} ||\hat{H}_n||_2^2$

Unitary transformation simplifies H_2 norm evaluation 2(2)

H_2 norm evaluated as sum of subsystem norms: $||H||_2^2 = \sum_{n=2}^{N} ||\hat{H}_n||_2^2$

- For n such that $\hat{\mathcal{A}}_n$ Hurwitz, $||\hat{H}_n||_2^2 = \mathrm{tr}\left(\hat{\mathcal{B}}_n^* X_n \hat{\mathcal{B}}_n\right)$, where

$$\hat{\mathcal{A}}_n^* X_n + X_n \hat{\mathcal{A}}_n = -\hat{\mathcal{C}}_n^* \hat{\mathcal{C}}_n. \quad (1)$$

Example:

Assume $\hat{\mathcal{A}} = -\mathrm{diag}\{\lambda_n^{\mathcal{A}}\}$, then $\hat{\mathcal{A}}_n = -\lambda_n^{\mathcal{A}}$ is scalar. Here, $\hat{\mathcal{B}}_n = 1$.

Then (1) gives $$||\hat{H}_n||_2^2 = X_n = \frac{\hat{c}_n^2}{2\lambda_n^{\mathcal{A}}}$$

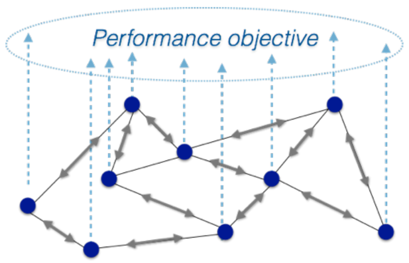
=1 with given output

Eigenvalue of $\mathcal{A}$!

so $||H||_2^2 = \frac{1}{2}\sum_{n=2}^{N} \frac{\hat{c}_n^2}{\lambda_n^{\mathcal{A}}}$.

H_2 norms involve sums over inverted eigenvalues, $\sum_{n=2}^{N} \frac{1}{\lambda_n^{\mathcal{A}}}$!

OUTLINE



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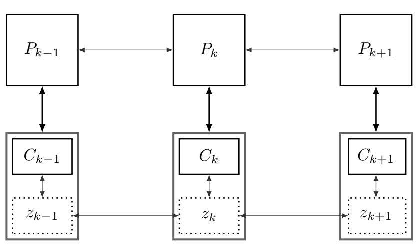
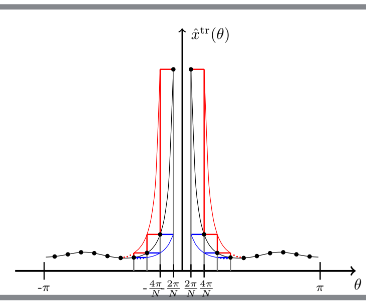


Problem setup: static vs. dynamic feedback



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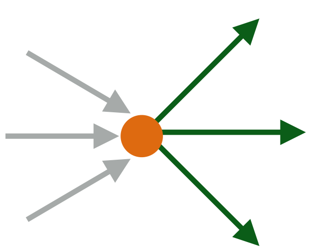
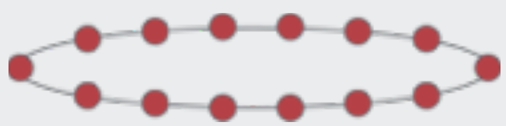


Conclusions and future work

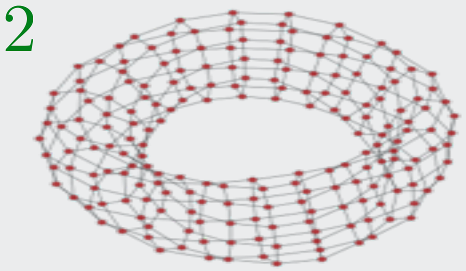
Consensus and vehicular formation problems modeled over toric lattices

- Network structure: d-dimensional discrete torus $\mathbb{Z}_N^d$. Network size: $M = N^d$.

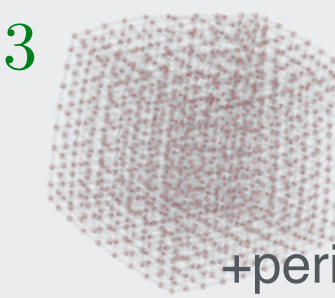
$d = 1$



$d = 2$



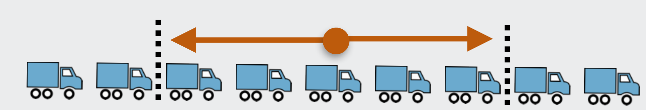
$d = 3$



  - *Spatial invariance!*

- Dynamics:
  - Consensus (1st order) $\dot{x}_k = u_k + w_k$
  - Vehicular formations (2nd order) $\ddot{x}_k = \dot{v}_k = u_k + w_k$

- Control: Standard, static feedback
  - Consensus: $u_k = (Fx)_k$
  - Vehicular formations: $u_k = (Fx)_k + (Gv)_k$
  - F, G are spatial convolution operators *(look like graph Laplacians / Toeplitz matrices)*



Only feedback from *local* neighborhood!

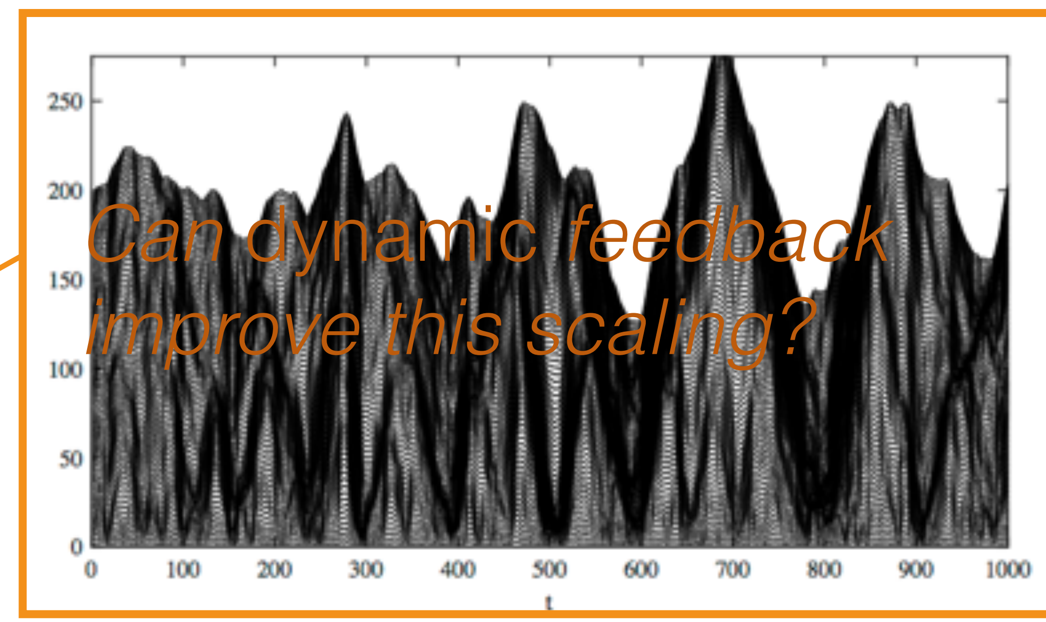
With static feedback: performance scales badly in low lattice dimensions

- Study *asymptotic* scaling of $V_k = \mathbb{E}\left\{\left(x_k - \frac{1}{M}\sum_{j\in\mathbb{Z}_N^d} x_j\right)\right\} = \frac{1}{M}||H||_2^2$
- The better V_k scales, the more *coherent* the system
- *Fully* coherent if V_k does not grow as network size $M \to \infty$

<u>Asymptotic scalings with *static* feedback (Bamieh *et al.*, 2012)</u>

Up to a constant independent of gain parameter β and network size M

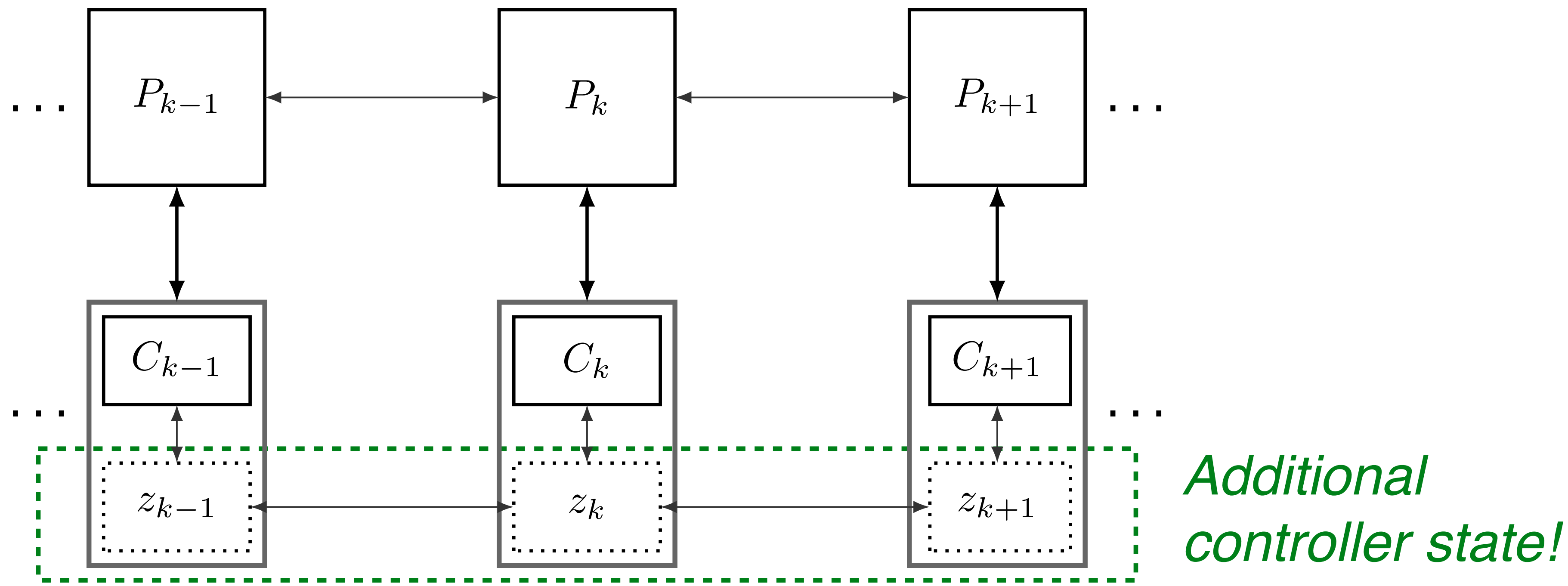
Absolute x, v	$V_k \sim \frac{1}{\beta}$
Relative x, absolute v	$V_k \sim \frac{1}{\beta}\begin{cases} M & d=1 \\ \log M & d=2 \\ 1 & d\geq 3,\end{cases}$
Relative x, relative v	$V_k \sim \frac{1}{\beta^2}\begin{cases} M^3 & d=1 \\ M & d=2 \\ M^{1/3} & d=3 \\ \log M & d=4 \\ 1 & d\geq 5,\end{cases}$



$$u_k = f_+(x_{k+1}-x_k) + f_-(x_{k-1}-x_k) + g_+(v_{k+1}-v_k) + g_-(v_{k-1}-v_k) + f_o x_k + g_o v_k$$

Relative feedback from x — Relative feedback from v — Absolute feedback

Introducing distributed *dynamic* feedback: control with memory



- Proposed control: General dynamic feedback
  - Consensus:

$$u_k = (Fx)_k + z_k$$
$$\dot{z}_k = (Az)_k + (Bx)_k$$

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} A & B \\ I & F \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w$$

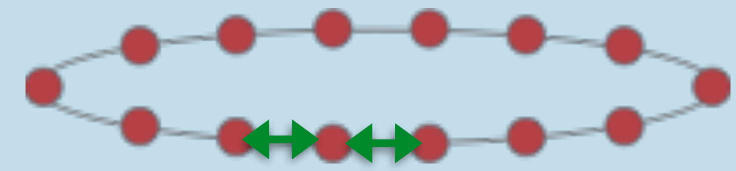
  - Vehicular formations:

$$u_k = (Fx)_k + (Gv)_k + z_k$$
$$\dot{z}_k = (Az)_k + (Bx)_k + (Cv)_k$$

$$\begin{bmatrix} \dot{z} \\ \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A & B & C \\ 0 & 0 & I \\ I & F & G \end{bmatrix} \begin{bmatrix} z \\ x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} w$$

Bad performance scaling is caused by eigenvalues near zero

Example (Standard consensus, 1st order):



$$\dot{x} = Fx + w$$

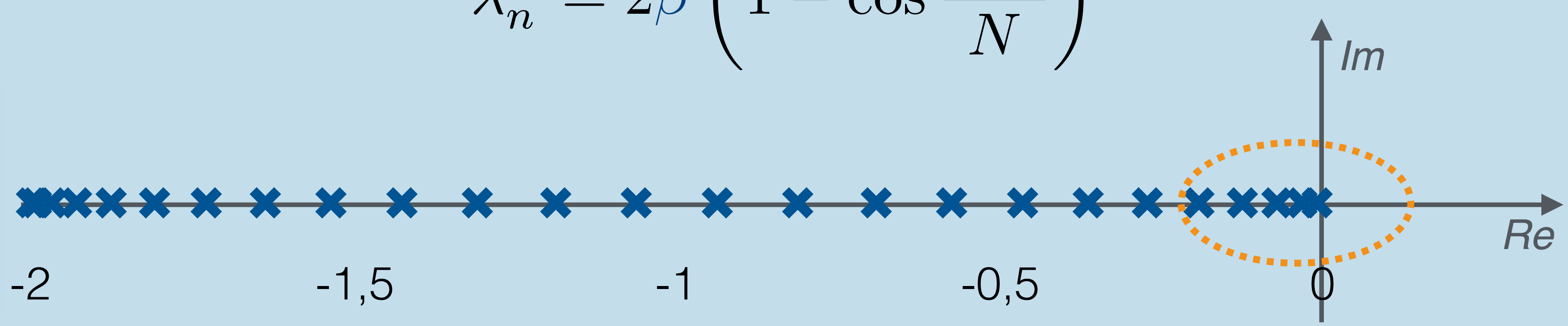
$$\dot{x}_k = \beta(x_{k+1} - x_k) + \beta(x_{k-1} - x_k) + w_k$$

Recall:

$$V_k = \frac{1}{N}||H||_2^2 = \frac{1}{2N}\sum_{n=2}^{N}\frac{1}{\lambda_2^F}$$

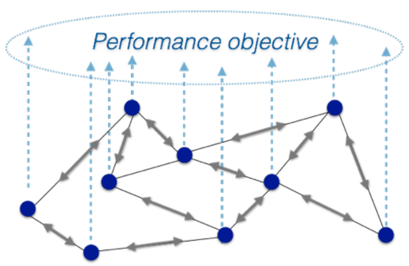
- Eigenvalues

$$\lambda_n^F = 2\beta\left(1 - \cos\frac{2\pi n}{N}\right)$$



- As N grows: Arbitrarily many λ_n^F increasingly close to zero - sum blows up!
- Need 3 spatial dimensions ($d = 3$) for sum to not scale worse than $M = N^d$.

OUTLINE



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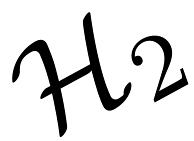

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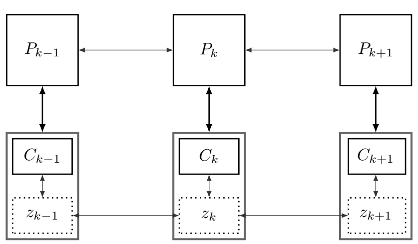

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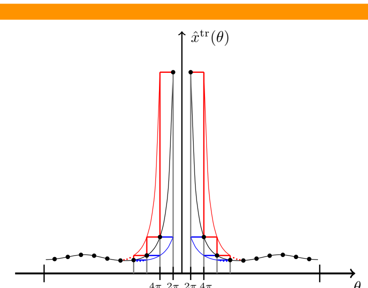
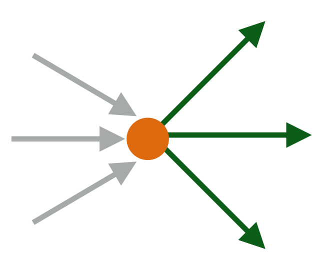


Deriving asymptotic performance scalings



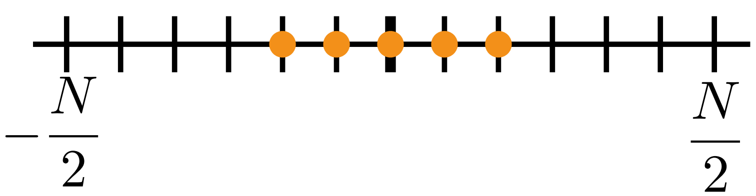
Conclusions and future work

Evaluating performance in the limit from finite to infinite lattices 1(2)

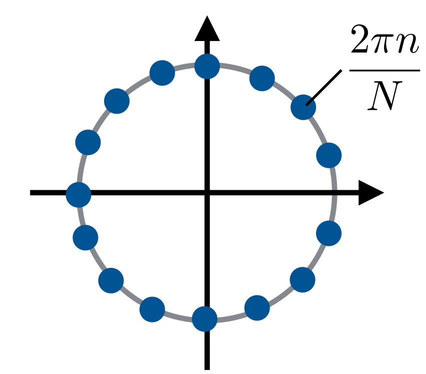
Recall:

Feedback operators are circular convolution operators on $\mathbb{Z}_N^d$,
for example $u_k = (Fx)_k = \sum_{l \in \mathbb{Z}_N^d} f_{k-l} x_l$

- Eigenvalues λ_n^F ⟷ Spatial discrete Fourier transforms

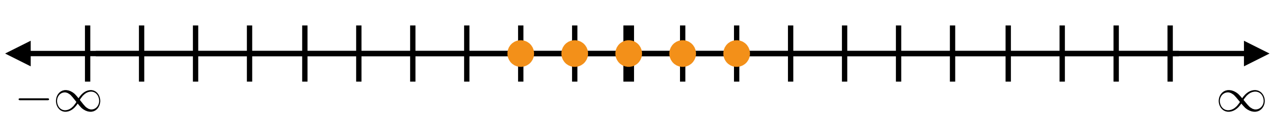


$$\hat{f}_n := \sum_{k \in \mathbb{Z}_N^d} f_k e^{-j \frac{2\pi}{N} n \cdot k}$$

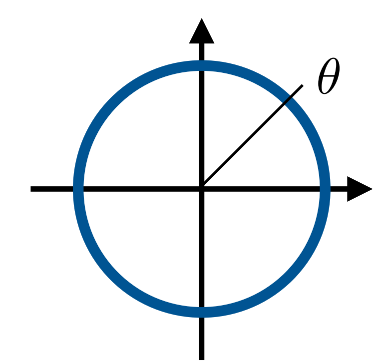


In limit of infinite lattice:

- ⟷ Z-transforms



$$\hat{f}_\infty(\theta) := \sum_{k \in \mathbb{Z}^d} f_k e^{-j \theta \cdot k}$$

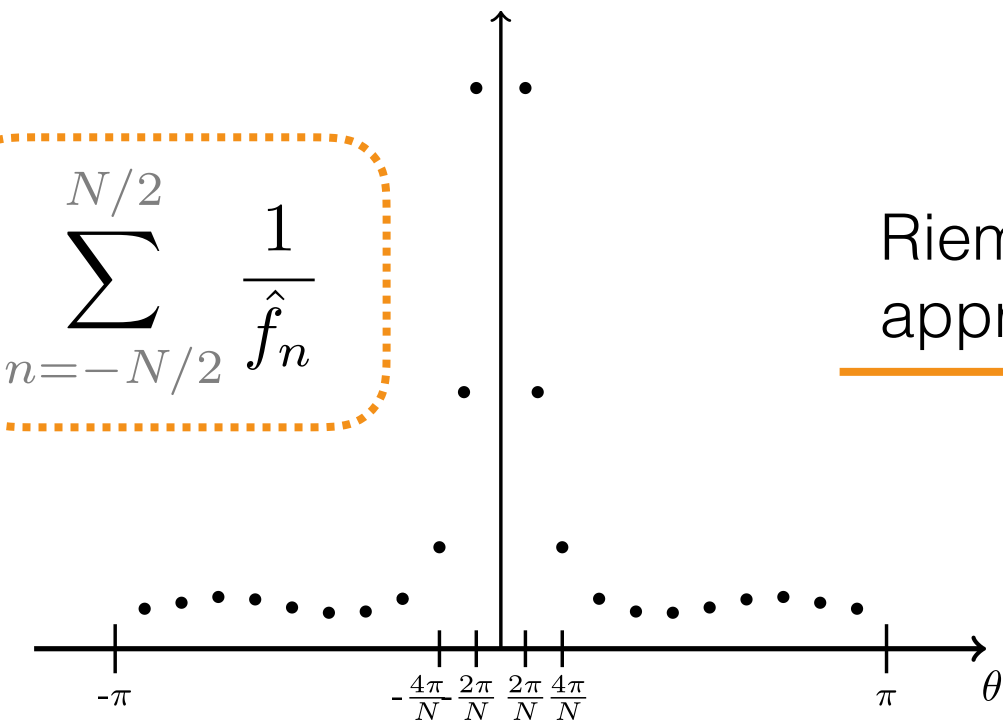


$$\hat{f}_n = \hat{f}_\infty\left(\frac{2\pi}{N} n\right)$$

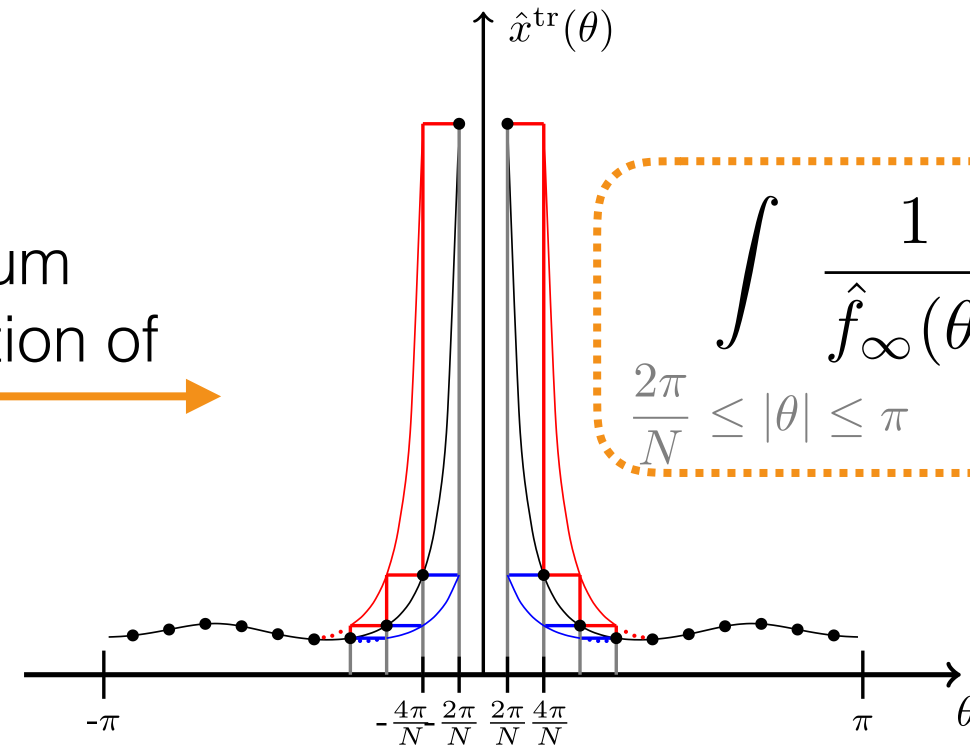
Evaluating performance in the limit from finite to infinite lattices 2(2)

- H_2 norm is a sum over subsystem norms (traces of Gramians)
- *Idea:* Estimate sum using integral

$$\sum_{n=-N/2}^{N/2} \frac{1}{\hat{f}_n}$$



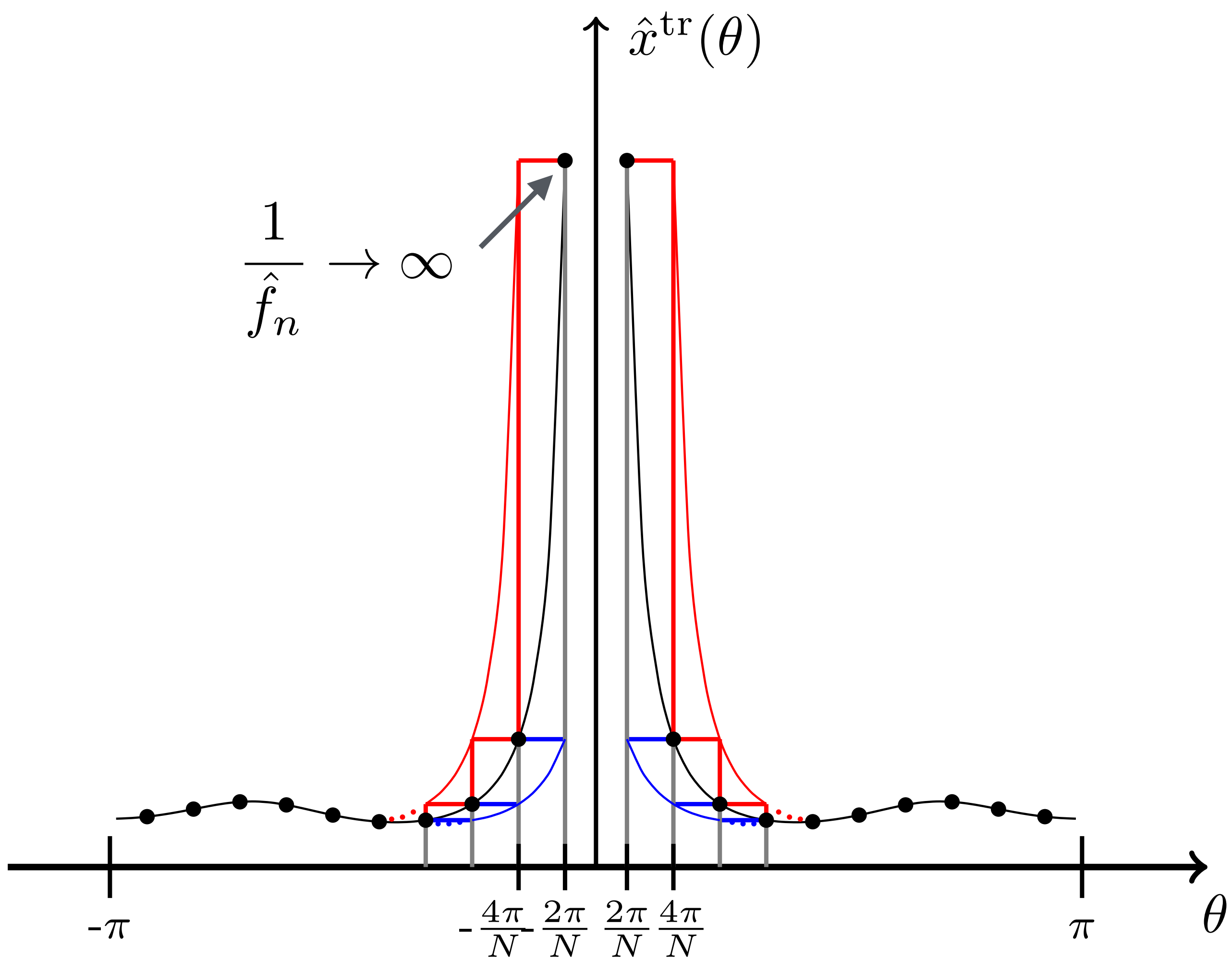
Riemann sum approximation of →



$$\int_{\frac{2\pi}{N} \le |\theta| \le \pi} \frac{1}{\hat{f}_\infty(\theta)} \mathrm{d}\theta$$

- See paper for technical details
- Holds for all $N > \bar{N}$, for some fixed $\bar{N}$, (hence "*asymptotic* scalings").

Asymptotic performance scaling is determined by *how fast* Gramian "blows up"



- Sum estimated through integral like

$$\int_{\frac{2\pi}{N} \le |\theta| \le \pi} \frac{1}{\hat{f}_\infty(\theta)} \mathrm{d}\theta$$

$\hat{x}^{\mathrm{tr}}(\theta)$ *"Gramian fctn"*

- Typically, $\hat{x}^{\mathrm{tr}}(\theta)$ has singularity at zero
- Order of singularity p determines scaling of H_2 norm in N

Lemma

If $\hat{x}^{\mathrm{tr}}(\theta) \sim \frac{1}{|\beta\theta|^p}$, then $V_k \sim \frac{1}{\beta^{p/2}} \begin{cases} N^{p-d} & \text{if } d < p \\ \log N & \text{if } d = p \\ 1 & \text{if } d > p \end{cases}$

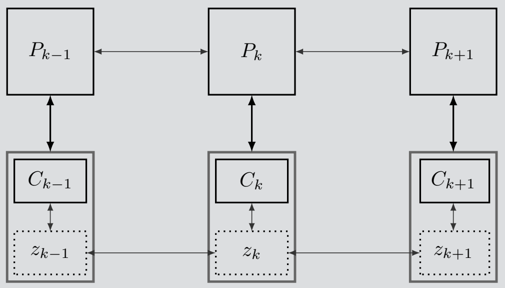
(Notation $u(x) \sim v(x)$ means $u(x)/v(x)$ uniformly bounded)

Dynamic feedback does *not* change scaling - if feedback is *relative*

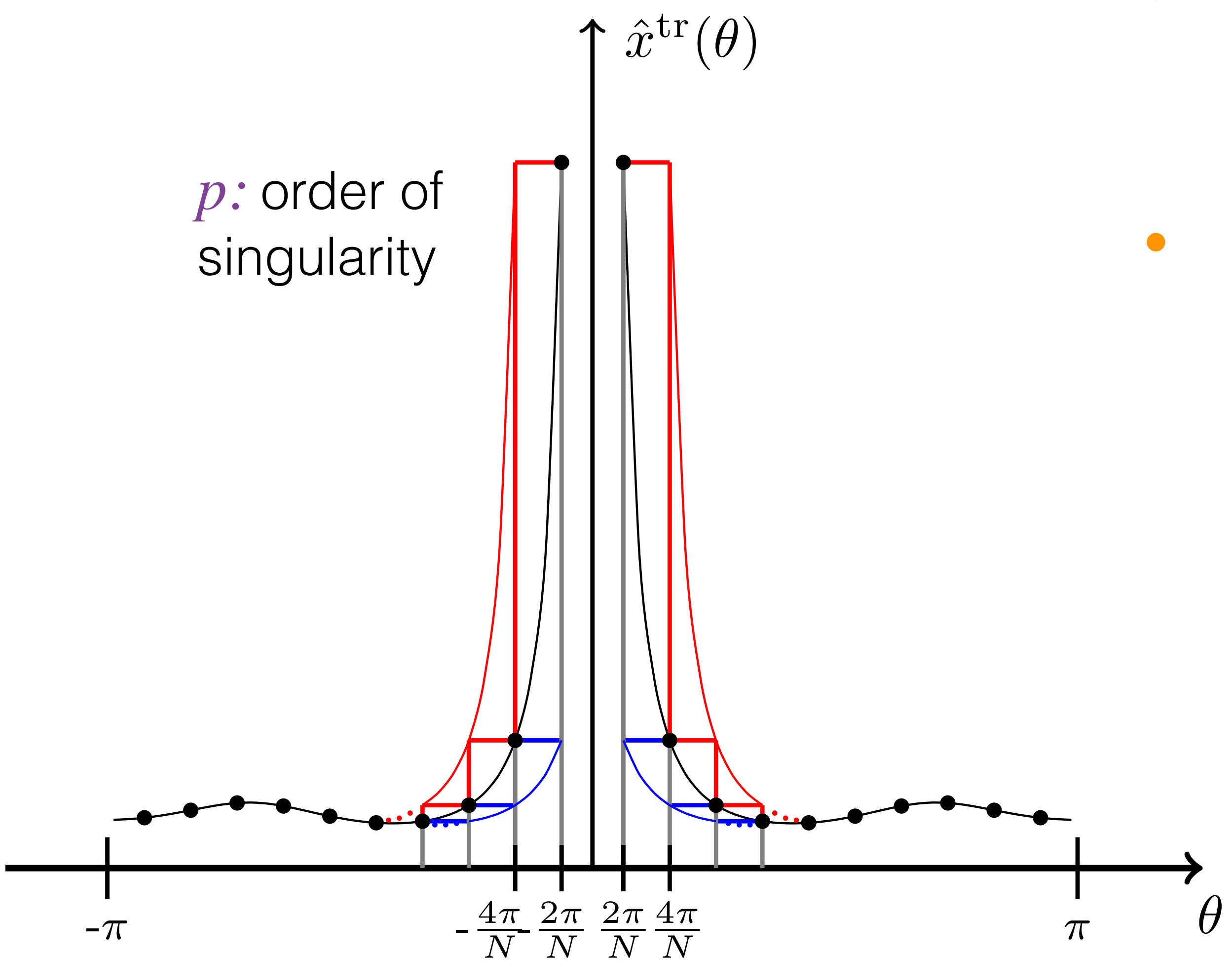
Recall: dynamic feedback laws

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} A & B \\ I & F \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w$$

$$\begin{bmatrix} \dot{z} \\ \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A & B & C \\ 0 & 0 & I \\ I & F & G \end{bmatrix} \begin{bmatrix} z \\ x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} w$$



p: order of singularity



- Only relative feedback from x, v
  - F, G, B, C all have zero eigenvalues!
- Scaling of Gramian unchanged
  - Consensus

$$\hat{x}^{\mathrm{tr}}(\theta) = \frac{-1}{2\hat{f}(\theta) + 2\varphi^c(\theta)} \sim \frac{1}{|\beta\theta|^2} \quad p = 2$$

From dynamic feedback

  - Vehicular formations

$$\hat{x}^{\mathrm{tr}}(\theta) = \frac{d}{2\hat{f}(\theta)\hat{g}(\theta) + 2\varphi^v(\theta)} \sim \frac{1}{|\beta\theta|^4} \quad p = 4$$

From dynamic feedback

Performance can be improved if absolute feedback available

<u>Asymptotic performance scalings for the vehicular formation problem:</u>

	Static feedback	Dynamic feedback	
Absolute x, v	$V_k \sim \frac{1}{\beta}$		*Same!*
Relative x, absolute v	$V_k \sim \frac{1}{\beta}\begin{cases} M & d=1 \\ \log M & d=2 \\ 1 & d \geq 3, \end{cases}$	$V_k \sim \frac{1}{\beta}$ *With perfectly noise-less measurements!*	*Improved performance!*
Relative x, relative v	$V_k \sim \frac{1}{\beta^2}\begin{cases} M^3 & d=1 \\ M & d=2 \\ M^{1/3} & d=3 \\ \log M & d=4 \\ 1 & d \geq 5, \end{cases}$		*Same!*

- Dynamic feedback with absolute velocities can give "absolute position feedback": *improves performance!*

$$u_k = \text{relative feedback} - g_o v_k + z_k$$

$$\dot{z}_k = a_+(z_{k+1} - z_k) + a_-(z_{k-1} - z_k) - c_o v_k$$

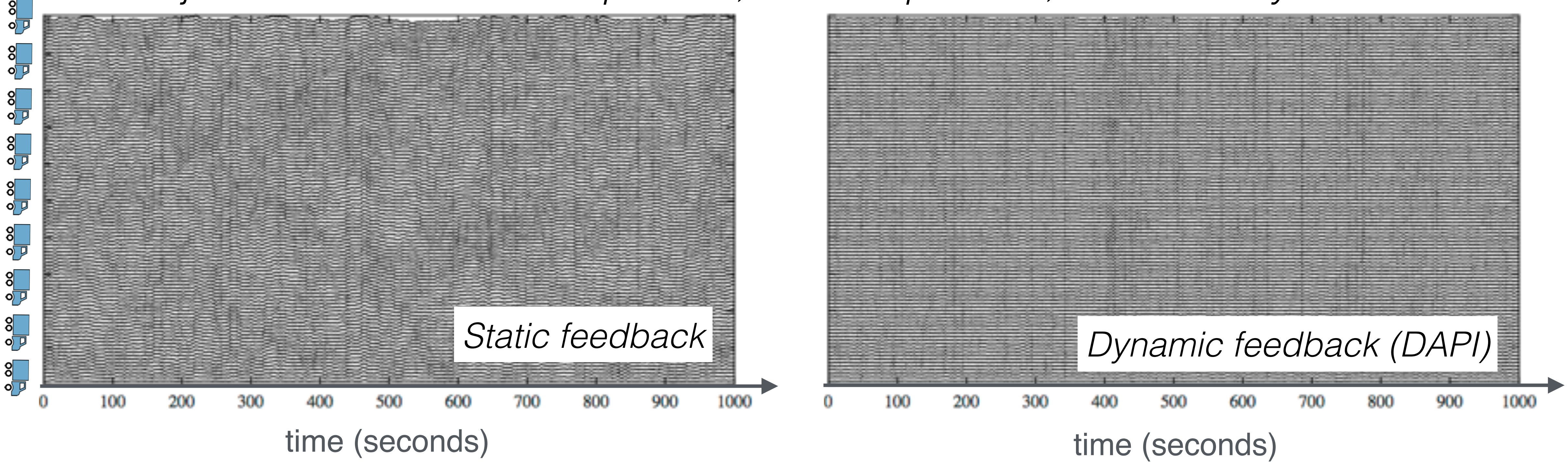
Distributed averaging of z

*Distributed averaging PI control **(DAPI)***

- With noisy measurements - distributed averaging of memory states needed

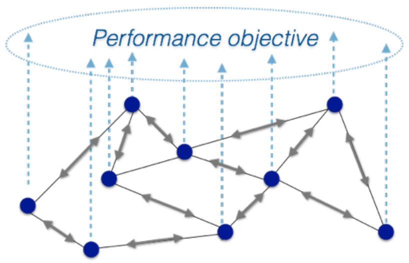
DAPI improves performance if absolute velocity measurements are available

Time trajectories of 100 vehicle platoon, with rel. position, abs. velocity feedback



- With noise: cannot achieve full coherence
- Still, performance improvement if noise small
- Useful if speedometers available, but absolute position unknown
- Also improves performance in power networks (ET *et al.*, ACC 2016)

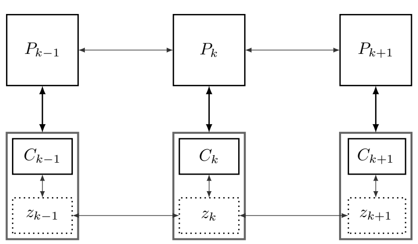
OUTLINE



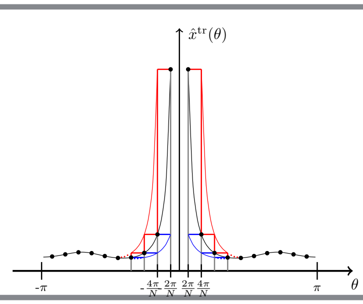
Introduction and problem overview

$\mathcal{H}_2$

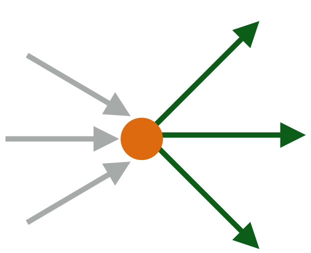
Evaluating input-output performance



Problem setup: static vs. dynamic feedback



Deriving asymptotic performance scalings



Conclusions and future work

Summary: Dynamic feedback does *not* achieve full coherence in low spatial dimensions

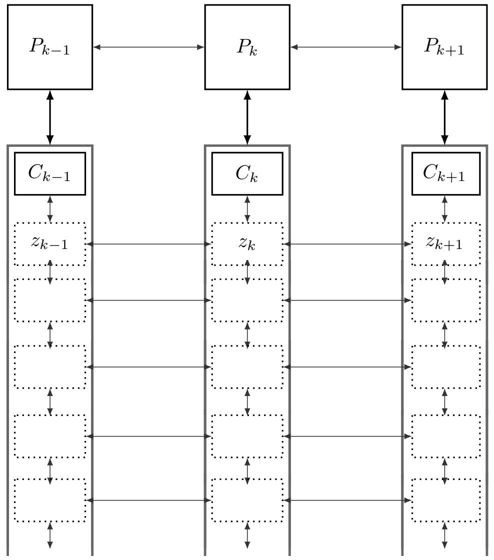
- Objective: Can dynamic feedback improve coherence of large-scale consensus and vehicular formation systems?
- Analyzed using spatial Fourier transforms, in limit of infinite lattice
- No performance improvement with only *relative* state feedback
- If *absolute* velocity feedback available: distributed PI control improves performance, but only ideally achieves full coherence in 1D
- (*Not covered:* many dynamic feedback laws unstable for large networks)

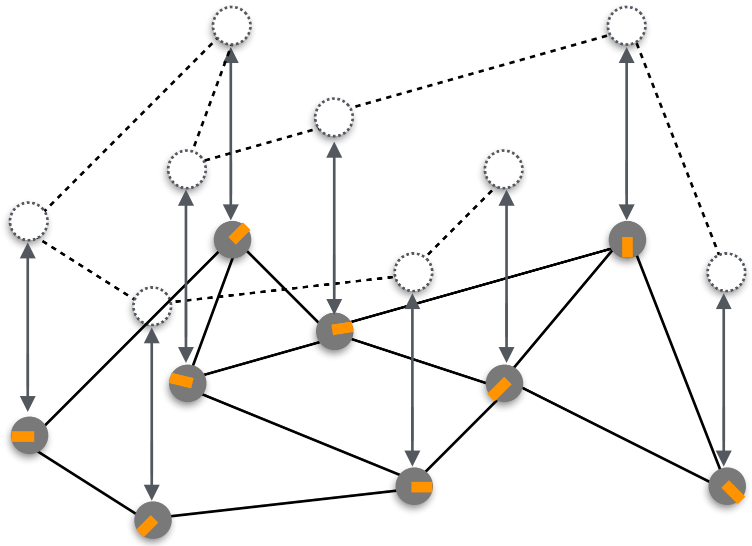
See also: E. Tegling: *On performance limitations of large-scale networks with distributed feedback control*, Licentiate thesis, KTH, May 2016

Future work includes further exploration of distributed dynamic feedback

<u>Topics to explore</u>

- Higher-order controllers
- DAPI control architecture
- Other performance metrics





$\mathcal{H}_2$ vs. $\mathcal{H}_\infty$ vs. …

Thank you!

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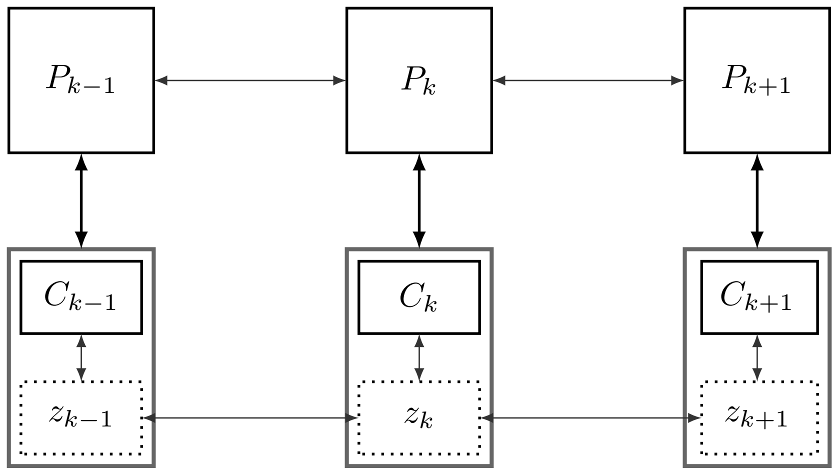
Additional material

Stability criteria: 1st order consensus with dynamic feedback

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} A & B \\ I & F \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w \qquad (1a)$$

$$y = \begin{bmatrix} 0 & I - \frac{1}{M} J_{\mathbf{1}} \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \qquad (1b)$$



Theorem

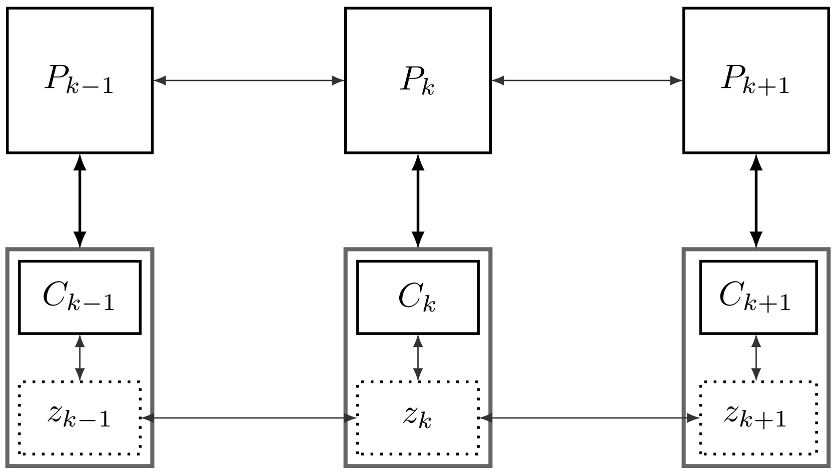
The system (1a) with fixed feedback operators A, B, F can be input-output stable with respect to the output (1b) for any lattice size N *only if:*

A. The operator B is symmetric around each network site*, or*
B. The operator A involves absolute feedback
or both.

Stability criteria: vehicular formation dynamics with dynamic feedback

$$\begin{bmatrix} \dot{z} \\ \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A & B & C \\ 0 & 0 & I \\ I & F & G \end{bmatrix} \begin{bmatrix} z \\ x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} w \quad \text{(2a)}$$

$$y = \begin{bmatrix} 0 & I - \frac{1}{M} J_{\mathbf{1}} \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \quad \text{(2b)}$$



Theorem 1 (conditions with only relative state feedback)

Assume that only relative measurements of x and v are available.

The system (2a) can be input-output stable w.r.t. the output (2b) for any lattice size N *only if*:

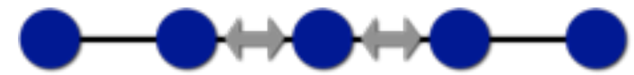
A. The operator $B = \mathbf{0}$, while A is non-zero, *or*

B. The operator A involves absolute feedback, *or both.*

Theorem 2 (integral control with absolute position feedback)

Assume that B involves absolute feedback from x. Then, a necessary condition for stability is that there is also absolute feedback from v.

Example: Explicit asymptotic performance of standard consensus algorithm.

- Standard consensus algorithm in 1D: 

$$u_k = \tilde{f}[(x_{k-1} - x_k) + (x_{k+1} - x_k)]$$

- or $u_k = \sum_{l \in \mathbb{Z}_N} f_{k-l} x_l$
  with $f_0 = -2\tilde{f},\ f_1 = f_{-1} = \tilde{f},\ \text{and } f_k = 0 \text{ for } |k| > 1$
- The Z-transform becomes: $\hat{f}_\infty(\theta) = \tilde{f}(-2 + e^{j\theta} + e^{-j\theta}) = -2\tilde{f}(1 - \cos\theta)$
- The Lyapunov equation gives the Gramian:

$$\hat{x}^{\mathrm{tr}}(\theta) = \frac{1}{2}\frac{-1}{\hat{f}_\infty(\theta)} = \frac{1}{\tilde{f}}\frac{1}{1-\cos\theta}$$

- The integral becomes:

$$I(\Delta) = \frac{1}{4\tilde{f}} \int_{\Delta \le |\theta| \le \pi} \frac{1}{1-\cos\theta} \mathrm{d}\theta$$

- Evaluated at $\Delta = \frac{2\pi}{N}$:

$$I\left(\frac{4\pi}{N}\right) = \frac{-1}{2\tilde{f}}\left[\cot\frac{\theta}{2}\right]_{\frac{2\pi}{N}}^{\pi} = \frac{1}{2\tilde{f}}\cot\frac{\pi}{N}$$

- which scales as $N/\tilde{f}$ asymptotically.