Scalability and fragility in bounded-degree consensus networks

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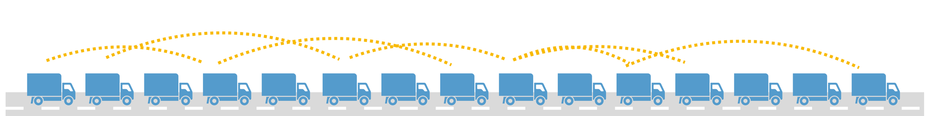
1 INTRODUCTION

Poor dynamic behaviors are often observed in large-scale network systems. In consensus networks, one example is scale fragility, where stability is lost for large networks[1,2]. Others are issues related to controllability and performance.

A network system's behavior should ideally be *scalable* — its performance should be uniform with respect to network size. At the same time, communications overheads should remain modest, imposing a uniform *bound on nodal degrees*.

For consensus networks, these objectives can be met by *expander families*. We show, however, that such networks are very fragile to *grounding*.

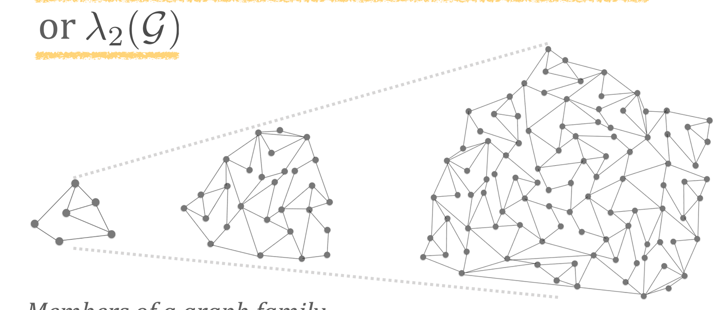


Vehicle platooning is one application where questions of scalability are relevant. Here, a communications infrastructure may add connectivity to improve performance.

2 PROBLEM SETUP

Network model

- Weighted, undirected, connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $|\mathcal{V}| = N$ nodes
- We consider *families* of graphs $\{\mathcal{G}_N\}$ in which N is increasing
- Graph Laplacian of $\mathcal{G}$ is denoted L, with eigenvalues $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$
- The *algebraic connectivity* of $\mathcal{G}$ is λ_2, or $\lambda_2(\mathcal{G})$



Members of a graph family

Consensus dynamics

- Each node is n^{th} order integrator
$$\tfrac{\mathrm{d}}{\mathrm{d}t} x_i^{(n-1)}(t) = u_i(t)$$
- Linear n^{th} order consensus:
$$u_i = -\sum_{k=0}^{n-1} a_k \sum_{j \in \mathcal{N}_i} w_{ij}(x_i^{(k)} - x_j^{(k)})$$
- Closed-loop dynamics:
$$\frac{\mathrm{d}}{\mathrm{d}t}\xi = \begin{bmatrix} 0 & I_N & 0 & \cdots & 0 \\ 0 & 0 & I_N & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_N \\ -a_0 L & -a_1 L & -a_2 L & \cdots & -a_{n-1} L \end{bmatrix} \xi$$

n = 1 gives information consensus $\dot{x} = -Lx$,
n = 2 gives vehicular formation dynamics $\ddot{x} = -a_0 Lx - a_1 L\dot{x}$.

Key assumptions

1. Bounded neighborhoods
$$|\mathcal{N}_i| \leq q \quad \forall i \in \mathcal{V}$$
2. Fixed* and finite weights
$$w_{ij} \leq w_{\max} < \infty$$
3. Fixed* and finite gains
$$a_k \leq a_{\max} < \infty$$

*independent of N

Leader-follower consensus is obtained by *grounding* one node; fix $x_1 \equiv 0$ and let $\bar{x} = (x_2, \ldots, x_N)$. Then, $\bar{x}^{(n)} = -a_o \bar{L}\bar{x} - a_1 \bar{L}\dot{\bar{x}} - \cdots - a_{n-1}\bar{L}\bar{x}^{(n-1)}$. $\bar{L}$ is the grounded Laplacian with smallest eigenvalue $\bar{\lambda}_1(\mathcal{G}) > 0$.

RECAP: THE ROLE OF ALGEBRAIC CONNECTIVITY

Convergence rate

The rate of convergence to consensus is inversely related to λ_2.
In 1st order consensus (n = 1):
$$\|x(t) - x^\star\| \leq \|x(0) - x^\star\| e^{-\lambda_2 t}.\ [3]$$

Sensitivity

Assume disturbance input $\dot{\xi} = A\xi + d$ and let $y = x - x^\star$. Then,
$$\|G_{d \to y}\|_\infty \geq \frac{1}{a_0 \lambda_2}$$
if n = 1 or n = 2 [4,5].

Stability

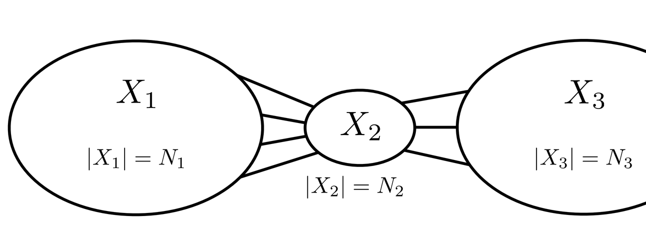
If n > 2, a necessary stability condition is
$$\lambda_2 > \frac{a_{n-3}}{a_{n-1}a_{n-2}}.\ [2]$$
If $\lambda_2 \xrightarrow{N\to\infty} 0$, the system is unstable for all $N > N^{\text{crit}}$.

In leader-follower consensus, these properties instead depend on $\bar{\lambda}_1$.

3 CONNECTIVITY SCALING AND EXPANDERS

The algebraic connectivity scales badly in N if the graph family has what we term a 'bottleneck'. Here, we characterize such families algebraically. Graph families without bottlenecks are termed *expander families.* Their algebraic connectivity scales well.

Graphs with decreasing algebraic connectivity



Graph partitioning for Lemma 1. The set X_2 is a 'bottleneck' if it stays small relative to X_1 and X_3.

- Partition a graph's vertices into three sets, so that X_2 is the *boundary set* of X_1 and X_3.
- The Laplacian becomes
$$L = \begin{bmatrix} L_1 & L_{12} & 0_{N_1 \times N_3} \\ L_{12}^T & L_2 & L_{32}^T \\ 0_{N_3 \times N_1} & L_{32} & L_3 \end{bmatrix}$$

Lemma 1: If every graph in the family $\{\mathcal{G}_N\}$ can be partitioned as above, so that $N_2/N_1 \to 0$ and $N_2/N_3 \to 0$ as $N \to \infty$, then
$$\lambda_2(\mathcal{G}_N) \xrightarrow{N\to\infty} 0.$$

Proof: Relies on Rayleigh-Ritz theorem.

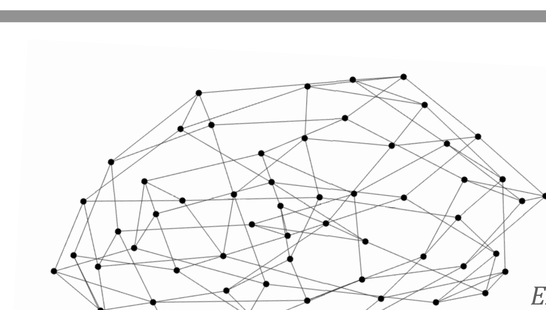
Expander families

- A graph's Cheeger constant is defined as
$$h(\mathcal{G}) = \inf_{X \subset \mathcal{V}} \frac{|\partial X|_d}{\min\{|X|_d, |V \backslash X|_d\}}$$

∂X Boundary set of X, $|X|_d := \sum_{i \in X} \sum_{j \in \mathcal{N}_i} w_{ij}$

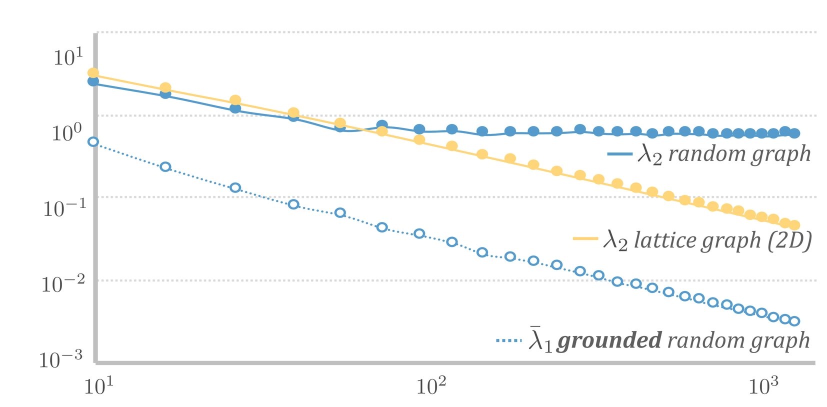
Definition: The graph family $\{\mathcal{G}_N\}$ is an *expander family* if $\{h(\mathcal{G}_N)\}$ is bounded away from zero as $N \to \infty$.

Result 2: The graph family $\{\mathcal{G}_N\}$ is an *expander family* if and only if $\{\lambda_2(\mathcal{G}_N)\}$ is bounded away from zero as $N \to \infty$. [6]



Example of randomly generated graph with N = 60.

- Random graphs are *almost surely* expander families. This can be exploited to construct consensus networks with good scalability[7].



Scaling of algebraic connectivity (and grounded Laplacian eigenvalue) in random graphs vs. lattice graphs. Nodal degrees are all 4 in both graphs. Note the difference in scaling between λ_2 and $\bar{\lambda}_1$ of the random graphs.

4 FRAGILITY TO NETWORK GROUNDING

Above, we saw that a good scaling of algebraic connectivity λ_2 is achieved by expander families. However, in grounded networks, the smallest eigenvalue of the grounded Laplacian, $\bar{\lambda}_1$ *inevitably* decreases towards zero as the network grows.

Lemma 2: Under Assumptions 1 and 2, the smallest eigenvalue of the grounded Laplacian satisfies
$$\bar{\lambda}_1(\mathcal{G}_N) \leq \frac{q}{N-1} w_{\max}.$$

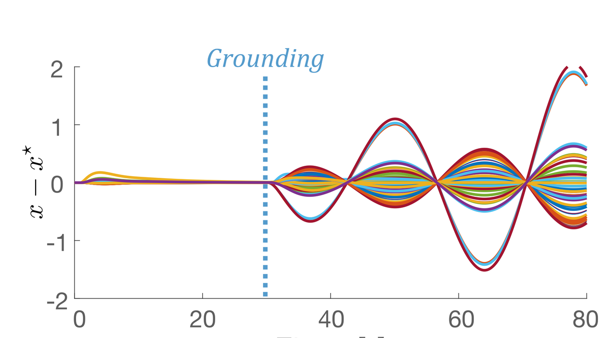
Proof: Relies on Rayleigh-Ritz theorem.

- Clearly, $\bar{\lambda}_1(\mathcal{G}_N) \xrightarrow{N\to\infty} 0.$

Implications

- Consensus is fragile to grounding — performance degrades severely if a node in an expander network is grounded. In high-order consensus, the system may destabilize.
- Leader-follower consensus (unlike leaderless consensus) *always* lacks scalability in bounded-degree networks.

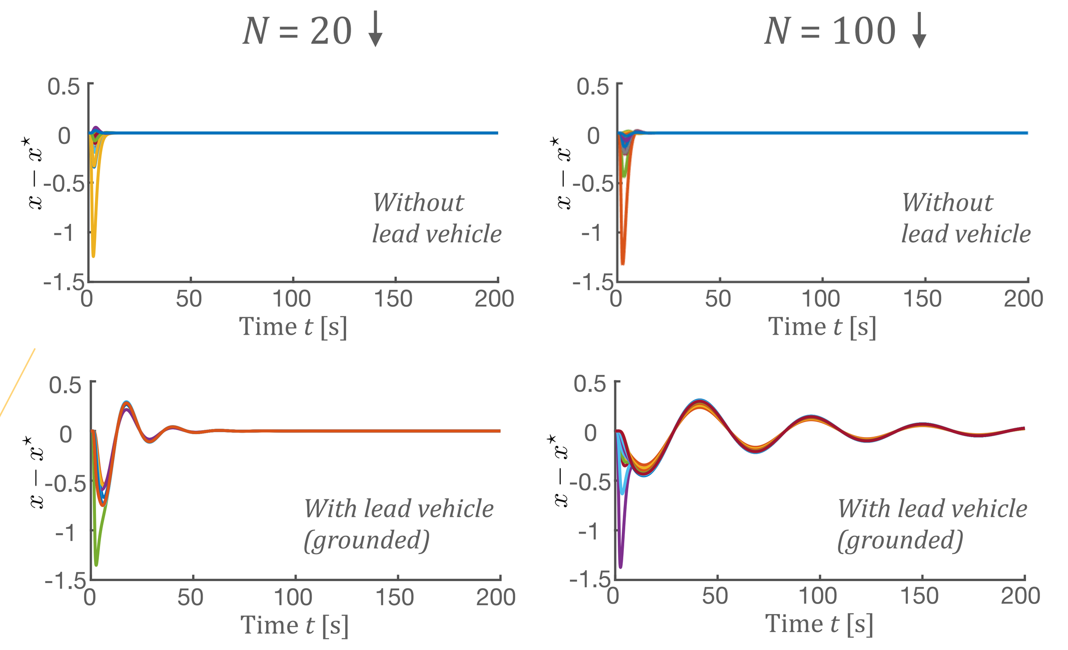
Example: The figure shows a simulation of 3rd-order consensus over a 60-node random graph. Node 1 is grounded at t = 30 s. This destabilizes the network, since $\bar{\lambda}_1 \ll \lambda_2$.



Fragility in n = 3 consensus network. The same disturbance is applied at t = 1 s and t = 31 s.

Example: A vehicular formation control problem is modeled by the consensus dynamics with n = 2. The formation's scalability can differ vastly depending on whether it has a lead vehicle.

Here, we simulate how small and large formations over random graph networks respond to a deceleration in one vehicle.



Simulation of formation control in, respectively, 20 and 100 node networks with and without lead vehicle. Scalability is lost if a lead vehicle is used.

5 KEY TAKEAWAYS

- It is desirable that the performance of a network system be *scalable* with respect to network size, though nodal degrees remain bounded.
- A high algebraic connectivity is crucial for the performance of consensus networks. It scales well only in expander families.
- Grounding the network creates leader-follower consensus, which is *never* scalable in bounded-degree networks.
- The results imply that large consensus networks can be highly fragile to grounding.

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